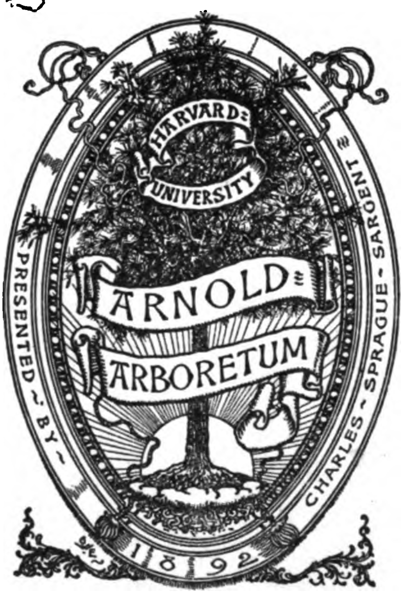
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VOLUME 8, 1922

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EDWIN BIDWELL WILSON, *Managing Editor*

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PROCEEDINGS
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Volume 8 DECEMBER 15, 1922 Number 12

PROJECTIVE AND AFFINE GEOMETRY OF PATHS

By O. Veblen

Department of Mathematics, Princeton University

Communicated, October 31, 1922

1. A path is defined as any curve given parametrically by a set of solutions of the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \quad \Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha} \tag{1.1}$$

The same system of paths may, however, be defined in terms of the same system of coördinates by another set of differential equations

$$\frac{d^2x^i}{dt^2} + \Lambda^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \quad \Lambda^i_{\alpha\beta} = \Lambda^i_{\beta\alpha} \tag{1.2}$$

The functions $\Gamma^i_{\alpha\beta}$ give rise to one definition of covariant differentiation and infinitesimal parallelism, and the functions $\Lambda^i_{\alpha\beta}$ to a different definition of these operations and relations. But both types of parallelism and of covariant differentiation refer to the same system of paths.

The theorems which state general properties of a system of paths, without restriction as to the scope of the theorems, constitute a projective geometry of paths. The theorems which state properties of the paths and of a particular set of functions, $\Gamma^i_{\alpha\beta}$, i.e., of a particular definition of infinitesimal parallelism, constitute an affine geometry of paths.

The discovery that there can be more than one affine geometry for a given system of paths is due to H. Weyl who also obtained an important tensor which he calls the *projective curvature*. Weyl's results are published in the *Göttinger Nachricten* for 1921 (p. 99), which has only recently reached this country.

The question whether there could be two sets of differential equations such as (1.1) and (1.2) for the same set of paths was raised by Professor Eisenhart (cf. this volume, p. 233) and answered in the negative under the

assumption that "the parameter s is the same for all the paths." The following paragraphs will serve to show wherein this assumption restricts the problem and also provide a second method of approaching the projective geometry of paths.

2. Let us first see how the differential equations (1.1) and (1.2) can give rise to the same system of paths. Any solution of the differential equations (1.1) can be written in the form (cf. p. 193 of this volume)

$$x^i = q^i + \psi^i (\xi^1 s, \xi^2 s, \ldots, \xi^n s) \tag{2.1}$$

in which $q^1, q^2, \ldots, q^n$ are the coördinates of an arbitrary point and $\xi^1, \xi^2, \ldots, \xi^n$ an arbitrary set of values of $dx^1/ds, dx^2/ds, \ldots, dx^n/ds$ at this point. The solution of (1.2) representing the same paths as (2.1) may be written in the form

$$x^i = q^i + X^i(\eta^1 t, \eta^2 t, \ldots, \eta^n t) \tag{2.2}$$

in which $\eta^1, \eta^2, \ldots, \eta^n$ is a set of values of $dx^1/dt, dx^2/dt, \ldots, dx^n/dt$ at the point q. By setting up a correspondence between the values of s and t which correspond to the same point of this path we define s as a function of t. This function depends on the point q and the direction of the path through q. Hence if $y^i = \eta^i t$, we may write

$$s = f(q^1, q^2, \ldots, q^n, y^1, y^2, \ldots, y^n) \tag{2.3}$$

for the value of s which corresponds to the same point as $q^1, q^2, \ldots, q^n$ and $y^1, y^2, \ldots, y^n$. As a transformation of the differential equation this may also be written.

$$s = f\left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dt}t, \frac{dx^2}{dx}t, \ldots, \frac{dx^n}{dt}t\right) \tag{2.4}$$

Let us now multilpy (1.1) by $(\partial f/\partial t)^2$ and add a term to each side so as to obtain

$$\frac{d^2x^i}{ds^2}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{dx^i}{ds}\frac{\partial^2 f}{\partial t^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\left(\frac{\partial f}{\partial t}\right)^2 = \frac{dx^i}{ds}\frac{\partial^2 f}{\partial t^2} \tag{2.5}$$

which is the same as

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \frac{dx^i}{dt}\Theta\left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}, t\right) \tag{2.6}$$

a differential equation in which t has exactly the same significance as in (1.2).

3. Let us now subtract (1.2) from (2.6). The result is

$$\varphi^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \Theta\frac{dx^i}{dt} \tag{3.1}$$

in which we are defining the functions $\varphi^i_{\alpha\beta}$ by the equations

$$\varphi^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} - \Lambda^i_{\alpha\beta} \tag{3.2}$$

The functions thus defined are the components of a tensor which is contravariant with respect to i and covariant with respect to α and β. This is because a transformation to new independent variables $z^1, z^2, \ldots, z^n$, changes the Γ's according to the formula

$$\Gamma^i_{\alpha\beta}(z) = \Gamma^j_{\gamma\delta}(x) \frac{\partial x^\gamma}{\partial z^\alpha} \frac{\partial x^\delta}{\partial x^\beta} \frac{\partial z^i}{\partial x^j} + \frac{\partial^2 x^j}{\partial z^\alpha \partial z^\beta} \frac{\partial z^i}{\partial x^j} \tag{3.3}$$

This tensor defines a covariant vector by means of the formula

$$\varphi_\alpha = \frac{1}{n+1} \varphi^i_{i\alpha} \tag{3.4}$$

By eliminating Θ from the equations (3.1) we obtain

$$\left(\varphi^i_{\alpha\beta} \frac{dx^j}{dt} - \varphi^j_{\alpha\beta} \frac{dx^i}{dt}\right) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0 \tag{3.5}$$

Remembering that the values of $dx^1/dt, dx^2/dt, \ldots, dx^n/dt$ are arbitrary it is easy to infer from (3.5) that

$$\varphi^i_{\alpha\beta} = \delta^i_\alpha \varphi_\beta + \delta^i_\beta \varphi_\alpha \tag{3.6}$$

where δ^i_α is 1 or 0 according as $i = \alpha$ or $i \neq \alpha$.

When (3.6) is substituted in (3.1) there results

$$2\varphi_\gamma \frac{dx^\gamma}{dt} = \Theta. \tag{3.7}$$

4. Conversely it can be shown that if we start with the differential equation (1.1) and any covariant vector φ_α, we can find a second differential equation of the type (1.2) which defines the same set of paths as (1.1). The first step is to define the functions $\varphi^i_{\alpha\beta}$ by means of equations (3.6). Since δ^i_α is a tensor and φ_α is a covariant vector it follows that $\varphi^i_{\alpha\beta}$ is a tensor of the third order. We then define $\Lambda^i_{\alpha\beta}$ by means of the equations (3.2) and write the differential equations (1.2) which are by definition identical with

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 2\varphi_\alpha \frac{dx^\alpha}{dt} \frac{dx^i}{dt} \tag{4.1}$$

If we compare (4.1) with (2.5) we see that (4.1) is obtainable from(1. 1) by means of the substitution

$$s = e^{2\varphi_\gamma \frac{dx^\gamma}{dt}} + h(x) \tag{4.2}$$

where $h\,(x)$ is an arbitrary function of $x^1, x^2, \ldots, x^n$.

It has been proved that: (1) If (1.1) and (1.2) are differential equations which define the same system of paths then the functions $\Gamma^{i}_{\alpha\beta}$ and $\Lambda^{i}_{\alpha\beta}$ are related by the equations

$$\Gamma^{i}_{\alpha\beta} - \Lambda^{i}_{\alpha\beta} = \varphi^{i}_{\alpha\beta} = \delta^{i}_{\alpha}\varphi_{\beta} + \delta^{i}_{\beta}\varphi_{\alpha} \tag{4.3}$$

in which φ_α is a vector. (2) If (1.1) are the differential equations of a system of paths and φ_α is any covariant vector then there exists a second set of differential equations (1.2) for the same system of paths, the functions $\Lambda^{i}_{\alpha\beta}$ being defined by (4.3).

The formula (3.7) reduces to that obtained by Eisenhart on page 235 of this volume in the case when s is a function of t alone. This makes it evident that his work must be taken as referring to a geometry in which there is an element of distance ds defined by means of functions of position. In this sense his theorem on page 235 will be found not to be in contradiction with the theorem (2) above.

A DIRECT METHOD OF TESTING COLOR VISION IN LOWER ANIMALS

By W. F. Hamilton

Department of Physiology, Yale University

Communicated, October 26, 1922

1. When spectral lights of different wave-lengths are directed on the photoreceptors of lower organisms, their reactions show that one of the two lights is, physiologically, the more effective. Even when the unequal distribution of energy in the spectrum is equated for, there still remains a difference in the physiological effectiveness of the stimuli. (Laurens and Hooker.[1]) This difference must be dependent upon wave-length. But it is also evident to the color blind eye (Koenig[2]). We cannot therefore argue that the differential response on the part of the organism is proof of color division.

Researches which have approached the question of color vision in lower organisms by means of *conditioned behavior* have not been done with pure spectral lights. Even when pure spectral lights are used, the difference in brightness must be equalized, not, however, in accordance with the luminosity curve of the human eye, but with that of the organisms whose color vision is to be tested. Obviously it is difficult to obtain a luminosity curve free from extraneous factors from the reactions of organisms sufficiently complex and intelligent to exhibit conditioned behavior.

The method of conditioned behavior is not applicable to lower organisms and it meets with difficulties when applied to higher animals. Therefore, if we would study the evolution of reactions to color, we must cast about for another method.

Most theories of color vision postulate receptive systems maximally sensitive to certain wave-lengths which differ in some manner from those which are maximally sensitive to other wave-lengths. The difference, according to some theories, lies in the receptive substance of the eye (Young-Helmholtz, Hering); according to others, there are different "color perceiving" mechanisms in the brain which serve for perception of the different fundamental colors. (Edridge Green,[3] Houstoun,[4] and the various "Zone Theories.") Work on the human eye (Burch,[5] Abney,[6] Frank Allen[7] and others) shows that these receptive systems can be separately fatigued by strong monochromatic light.

If we can get evidence that the photo receptors of lower organisms can be differentially fatigued in a similar manner, it would indicate that the receptive system which is maximally sensitive to certain wave-lengths differs in some respect from the receptive system which is maximally sensitive to other wave-lengths, and this could be construed into something analogous to color vision. The following is a preliminary report of the results of experiments recently performed with this in view.

2. The animal decided upon was the fruit-fly, Drosophila. Specimens were kept in the dark over night and used in the dark adapted state. The apparatus consisted of two Hilger wave-length spectrometers** used as monochromatic illuminators and so set as to throw beams of light into opposite ends of a small horizontal tubular glass cell containing the flies. The sources of light were two 500-watt Mazda C stereoptican lamps.

The intensities of the light beams, differing as to wave-length were so regulated that the animals were not oriented in either direction. The lights were physiologically equal. As soon as this was accomplished one beam of light, called for convenience the "stimulating light," was intercepted by a screen and the animals subjected to the action of the other ("fatiguing") light. As soon as a marked decrease in the positive reaction of the flies to the "fatiguing" light was observed, the screen was removed and the animals subjected to the "stimulating" light again.

As soon as this was done, the animals usually oriented and crawled toward the "stimulating" light in spite of the continued action of the "fatiguing" light. The operation of the "fatiguing" light had thus upset the physiological balance between the two lights by fatiguing the apparatus maximally sensitive to this light, more than the apparatus maximally sensitive to the "stimulating" light. When such a test had been complete, it was repeated immediately, with the light that had been used as

the "fatiguing" light arranged to "stimulate" and vice versa. In the second test, the animals were, of course, not dark adapted.

If, however, the two lights were of the same or nearly the same wave-lengths the results were quite different. The lights were balanced, the "stimulating" light intercepted, and the animals subjected to the "fatiguing" light. There was evidence of fatigue, but when the "stimulating" light was again allowed to fall on the flies, it was easily seen that the fatigue was not differential. The animals distributed themselves at random, between the two ends, just as they did when the lights were first balanced.

The spectrum was then scaled. The smallest difference in wave-length which showed conclusive evidence of differential fatigue was the unit used. Beginning at 385 $m\mu$, which is as low as the calibration of the instruments extend, the values were found as follows:

385–410, 410–430, 430–450, 450–500 $m\mu$

The last determination was rather doubtful. At the intensity available the orienting effect of light above 500 $m\mu$ was insufficient to give conclusive evidence of differential fatigue. The distribution of these points and the fact that "hue perception" is at its maximum between 410 $m\mu$ and 450 $m\mu$ makes it appear that there may be two receptor systems, one for the blue-violet and one for the blue-green. Evidence for a third component seems to be lacking. According to Steindler[8] the hue perception curve of the normal human eye indicates the existence of three receptor systems.

3. It is planned to extend this work to other organisms and to work out the energy—time relationships of differential fatigue with different pairs of wave-lengths at different intensities.

The method should prove of value in analyzing the wave-length receptors of lower organisms, and its modifications and applications are numerous. It is applicable to the investigation of the pupil reactions, of the action current of the optic nerve and to similar problems. Perhaps one of its most interesting applications would be to ascertain the course of evolution of the color sense among lower organisms by means of photic orientation in crossed beams of light. The chief value of the method is its objectiveness.

[1] Laurens, H., and Hooker, H. O., 1920, Studies on the relative physiological value of spectral lights. *J. Exper. Zoöl.*, **30,** 345–68.

[2] Koenig, *Ges. Abhandlungen*, Berlin.

[3] Edrige Green, F. W., 1920, *The Physiology of Vision*. London.

[4] Houstoun, R. A., 1916, A theory of color vision. *Proc. Roy. Soc. London*, **A92,** 424–32.

[5] Burch, G. T., 1899, On artificial temporary color blindness with an examination of the color sensation of 109 persons. *Proc. Roy. Soc.*, **B191,** 1–34.

[6] Abney, Sir W. de W., 1913, *Researches in Color Vision*. London.

[7] Allen, Frank, 1920, Persistence of vision and the primary color sensations. *Amer. J. Physiol. Optics*, 1, 94–134.

[8] Steindler, 1906, Der Farbenempfindlichkeit des normalen und farbenblinden Auges. *Sitz. Weiner Akad.*, 105, 11a, 115–6.

** One of these instruments belongs to this Laboratory, while the other was loaned to us by the kindness of Professor G. H. Parker, Harvard University.

CHARACTERISTICS OF A SHORT WAVE OSCILLATOR AT VERY LOW PRESSURES

By L. L. Nettleton

Department of Physics, University of Wisconsin

Communicated October 11, 1922

Various investigators have experimented with three element vacuum tubes oscillating at a wave-length of the order of one meter. All of them have used commerical tubes, having a cylindrical arrangement of filament, grid, and plate.

Barkhausen and Kurz[1] discovered a type of oscillation which was apparently due to the motion of electrons in the tube itself, and was independent of the inductance and capacity of the external circuit. The computed time for an electron to travel across the tube, from filament to plate and back, under the potentials applied, agreed fairly well with the period of the waves as measured by means of Lecher wires coupled to the tube.

Whiddington[2] experimented with a similar type of oscillation, but he observed much lower frequencies and attributed the observed effects to the motion of ions instead of electrons. An explanation was given which involved a discontinuous emission from the filament.

Gill and Morrell[3] have experimented extensively with the Barkhausen type of oscillation, using commercial tubes made by the Marconi company, and have given an explanation of the effect which involves a natural mode of oscillation of the electrical system connected to the tube.

The work of Barkhausen and Kurz and that of Gill and Morrell was done with so-called "hard" tubes, which in the process of manufacture have been thoroughly heated to remove occluded gas and then sealed off from the vacuum pump. These men have assumed that they were dealing with a purely electronic phenomenon, and that no gaseous ionization was concerned.

The essential departure in the experimental arrangement in the present investigation was the use of a tube which was left permanently connected to the vacuum pumps, and was made with a ground joint to make the in-

side parts accessible for modification, replacement of burned out filaments, etc. Also, more power was used than by other investigators, electron currents as high as 300 milamperes and at voltages up to 700 being used at times.

The electrical connections were made as indicated in figure 1, except that the filament heating circuit is not shown. The Lecher wires, LL, were closely coupled to, but not in metallic contact with, the grid and plate leads, which were brought straight in through the sides of the tube. A sliding bridge, B, across the Lecher wires, carried a crossed wire thermocouple which was connected to a sensitive galvanometer by long, twisted leads. This thermocouple served to measure the strength of oscillations, and by

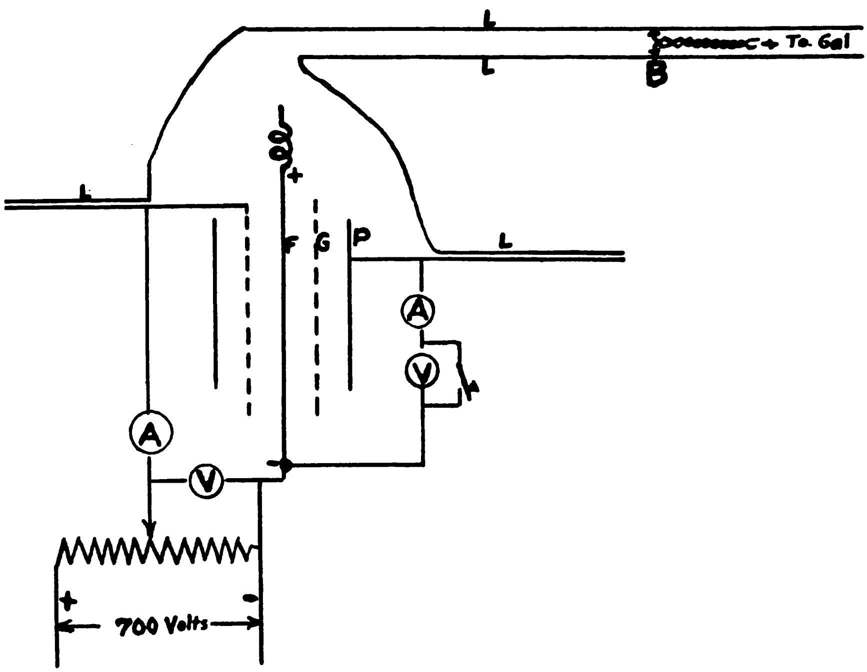

FIGURE 1

sliding the bridge along the wires, the points of maximum deflection gave the loops of the standing waves, and the distance between these points gave the half wave-length of the oscillations. In the discussion which follows, by strength of oscillations is meant the deflection of the thermocouple galvanometer when the bridge position is adjusted so that this deflection is a maximum.

Oscillations of the Barkhausen type, at wave-lengths of from 50 to 200 cm., were obtained. The unusual characteristic of a tube when oscillating in this way is that there is a negative current to the plate, which

may flow against a negative potential of 100 volts or more with respect to the filament, the grid being at a positive potential of several hundred volts. If the plate is simply connected to the filament through a voltmeter, this voltmeter may read as high as 150 volts, or if connected through an ammeter, the negative current to the plate may be as much as 25% of the total electron current from the filament. The magnitude of this negative current or voltage varies greatly with the temperature of the filament, grid voltage, and gas pressure. There is never oscillation in the Lecher wires without a negative current or potential on the plate, but this negative current may be present and yet the Lecher wires show very feeble or no oscillations. The characteristics of this negative current and the associated oscillations have been studied as affected by changes of grid voltage, of electron current, and of gas pressure.

It was found, very unexpectedly, that both the negative plate current and any sign of oscillation, as indicated by the Lecher wires, ceased abruptly

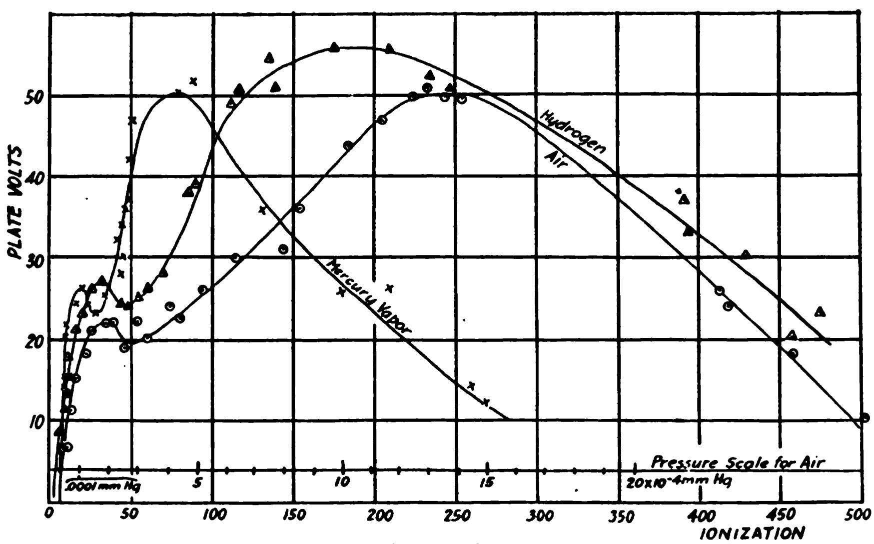

FIGURE 2

at very low pressures. This is shown in the curves of figure 2 for plate voltages, and in figure 3 for the strength of oscillation in the Lecher wires. These observations were all made with the grid at 250 volts and the filament heating currents such that the electron current was 200 milamperes. The pressure of the residual gas in the tube was measured with an ionization manometer. The pressures of hydrogen and air were controlled by adjusting the flame under the mercury vapor pump, with liquid air on the trap to keep out mercury vapor. The mercury vapor pressure was controlled by keeping the liquid air trap (with a pool of mercury at

the bottom) at different temperatures, the mercury vapor pump operating at its maximum rate to reduce the pressure of any gas to the minimum. The curves are plotted with the readings of the ionization manometer as abscissae. This was done because it is probable that the essential thing is not the presence of the gas itself, but of gaseous ions. Since different gases at the same pressure give different amounts of ionization when under the same ionizing agent, it seemed better to plot their ionization instead of pressure. The pressure corresponding to a given ionization current is

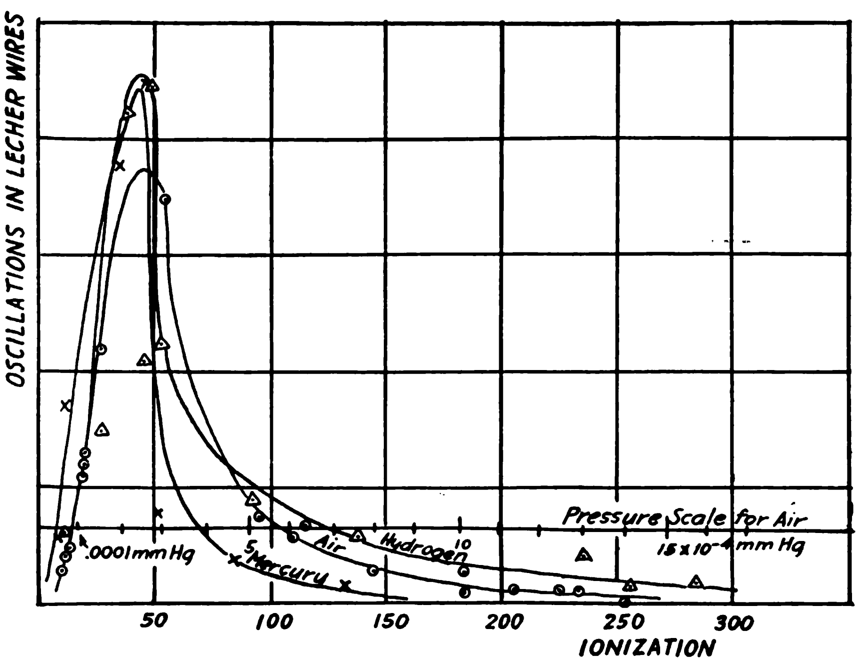

FIGURE 3

for mercury about half, and for hydrogen about twice that for air. The pressure scale for air is given on the curves, so by multiplying and dividing by 2 the approximate pressures of hydrogen and mercury, respectively, will be obtained.

These curves apparently indicate that a small amount of ionization is necessary for this type of oscillation. The kind of gas does not seem important. Since the gas pressure at which the oscillation stops is so low, being less than .00005 mm. there can be no connection with the mean free path of the ions, for at these pressures the path is many times longer than the dimensions of the tube. It is not likely that the commercial tubes used by other investigators are sufficiently exhausted to give any indication of the dropping off of the curves at the low pressure side. So far, no staisfac-

tory explanation has been found for the profound effect of such a small amount of gas or the behavior of the tube.

The investigation is being continued and a more detailed account will be published in the near future.

[1] Barkhausen and Kurz, *Phys. Zs., Leipzig*, **21**, No. 1, Jan. 1920, p. 1.

[2] Whiddington, *Radio Review*, Nov. 1919, p. 53.

[3] Gill and Morrell, *Phil. Mag.*, **44**, No. 259, July 1922, p. 161.

THE REFRACTION OF X-RAYS IN CALCITE

By Bergen Davis and H. M. Terrill

Department of Physics, Columbia University, N. Y.

Communicated, October 26, 1922

The effect of refraction in X-ray spectra has been discussed by Stenström,[1] who made some determinations of the index of refraction from the relative displacement of the several orders. In most cases the effect was too small to admit of measurement, but for sugar and gypsum crystals he obtained some values for wave-lengths greater than 2.5 Å.

The present paper applies a modification of the same method to the reflection from calcite of the $K\alpha_1$ line of Molybedenum, 70783 Å.

Since we measure the angle from the crystal face, and not the normal, the customary equation for the index of refraction becomes

$$\nu = \frac{\cos\theta}{\cos\theta'}, \tag{1}$$

where θ is the glancing angle outside the crystal and θ' the angle of the beam inside. We will use a subscript to indicate orders higher than the first. Placing $\nu = 1 - \delta$, Stenström computes the values of δ from the equation

$$\delta = \frac{\left(\frac{\sin\theta_m}{m}\right)^2 - \left(\frac{\sin\theta_n}{n}\right)^2}{2\left(\frac{\cos\theta_m}{m}\right)^2 - 2\left(\frac{\cos\theta_n}{n}\right)}$$

where m and n are any two orders.

As the shift is small in any case, it seemed desirable to express the value of δ directly in terms of angle, in order to more readily determine the effect of errors of observation which are liable to be of the same order of magnitude as δ itself. Now from (1) we have

$$\sin^2\theta' = 1 - \nu^{-2}\cos^2\theta$$

Since δ is small, we write

$$\nu^{-2} = 1 + 2\,\delta$$

whence

$$\sin\theta = [1 - \cos^2\theta\,(1 + 2\,\delta)]^{1/2} = [\sin^2\theta - 2\,\delta\cos^2\theta]^{1/2}$$

and again neglecting terms containing higher powers of δ.

$$\sin\theta' = \theta - \delta\,\frac{\cos^2\theta}{\sin\theta} \qquad (2)$$

Taking arc sin of each side, we have since δ is small,

$$\theta' = \theta - \delta\,\frac{\cos^2\theta}{\sin\theta}\cdot\frac{1}{\cos\theta} \qquad (3)$$
$$\theta' = \theta - \delta\cot\theta$$

Now in this equation, θ is the crystal angle measured, hence by determining the angles of any characteristic line for several orders, it will be possible to determine δ, or at least to find the limits of its magnitude. The angles being measured, those of higher orders are reduced to the equivalent first order angle, so that all angles may be compared directly. The proportion that each is affected by refraction being known from (3), the absolute value of δ is readily determined.

Instead of using (3), it is possible to substitute (2) in the relation

$$n\,\lambda = 2\,d\sin\theta_n'$$

obtaining, since δ is small

$$\frac{n\,\lambda}{2\sin\theta_n} = d\,(1 - \delta\cot^2\theta_n) \qquad (4)$$

thus determining δ from the increase in apparent grating space with order. However, since the limits of error have to be carried through in the computations, the previous method seems to be more convenient.

The spectrometer used has been described elsewhere.[2] The tube was of the water cooled type with a molybdenum target. Measurements of θ were made for the first three orders of the $K\alpha_1$ line, using a crystal of clear Iceland spar. The measurements were gone over several times and it was found that the angles could be repeated to within 15″. This is just about the limit of accuracy of the instrument which was determined by other methods to be about 20″.

The following mean values were obtained.

$$\theta = 6°\ 42'\ 43'' \pm 10''$$
$$\theta_2 = 13°\ 30'\ 45'' \pm 10''$$
$$\theta_3 = 20°\ 31'\ 22'' \pm 10''$$

where the additional terms represent the limits of error.

It should be remarked that if the crystal is not mounted so that the mean reflecting plane is directly over the center of rotation, and the face of the crystal be slightly curved, a shift of the lines will be obtained, that might be taken for refraction. To eliminate this possibility, the crystal was mounted on a slide with a micrometer screw attached, so that it might be moved in a direction perpendicular to its face. The crystal being mounted as nearly as possible over the center of rotation by mechanical methods, the adjustment was changed by small fractions of a millimeter, reading each time on both sides the crystal and chamber angles of the α_1 line. If curvature is present, the crystal angle will change as the crystal is moved. Twice the crystal angle is plotted against the readings of the micrometer head and on the same sheet, the chamber angle is also plotted against these readings. The intersection of the two lines gives the proper micrometer setting, since the crystal must be over the center, when the chamber angle is exactly twice the crystal angle. With the crystal used, it was found that the curvature was very slight, a shift of less than one second of arc being found within the range of adjustment. This was checked for each order.

We proceed then to reduce the angles measured to the corresponding first order angles. Now $\theta_n = \theta_n' + \Delta$ where Δ includes the slight increase due to refraction and also the error of observation. Since Δ is small

$$\sin\theta_n = \sin\theta_n' + \Delta\cos\theta_n$$

$$\frac{\sin\theta_n}{n} = \sin\theta_n' + \Delta\frac{\cos\theta_n}{n}$$

$$\text{arc}\sin\left(\frac{\sin\theta_n}{n}\right) = \theta' + \Delta\frac{\cos\theta_n}{n\cos\theta}$$

$$\theta' = \text{arc}\sin\left(\frac{\sin\theta_n}{n}\right) - \Delta\frac{\cos\theta_n}{n\cos\theta}$$

and the coefficient of Δ can be computed for each order, it being .49 for the second and .31 for the third. Multiplying the errror limits by these coefficients and introducing the refraction from (3) which is also multiplied by them, we have the following values for θ':

$$\begin{aligned}\theta' &= 6^\circ\ 42'\ 43'' \pm 10'' - 1''.76\cdot\delta\cdot 10^6\\ &= 6^\circ\ 42'\ 33'' \pm 5'' - 0''.63\cdot\delta\cdot 10^6\\ &= 6^\circ\ 42'\ 38''.^5 \pm 3'' - 0''.27\cdot\delta\cdot 10^6\end{aligned}$$

and we have to determine δ from these equations.

Plotting the values of the angles against the order, using lines covering the limits of error instead of points to locate the extremities of the ordinates,

we have to pass through these lines a curve whose ordinates are proportional to the coefficients of δ. Now the values found do not lie well on any curve of this type, the second order being entirely too low. The curves that fitted most nearly gave a value of $\delta = 3 \times 10^{-6}$ while the error limits would permit of values ranging from zero to several times this. It will be seen that this corresponds to a shift of the first order of 5″, so that for this wave-length, the effect of refraction is very slight.

This value $\delta = 3 \times 10^{-6}$ necessarily contains large possible errors. It is, however, of the same order of magnitude as that obtained by the method of total reflection.

An independent determination of δ was obtained by measuring the angle of total reflection as suggested by the experiments of A. H. Compton[3] on total reflection from glass and lead. The ionization chamber was shielded from the direct beam and the spectrum searched at angles close to grazing incidence. Total reflection appeared as a line, approximately 4′ wide at half maximum, slightly unsymmetrical and of an intensity about 25 times the surrounding values which coincided with the leak. To test if this line consisted principally of the monochromatic radiation corresponding to the $K\alpha$ line, a zirconium screen was introduced. This diminished the intensity without materially changing the shape of the line or shifting its position.

Measurements were made on each side of the zero position, and the mean of the readings gave 6′ 30″, the value being good to 30″. Now for total reflection, putting $\theta' = 0$ in (1) we have

$$\nu = \cos\theta \text{ or } \delta = \frac{\sin^2\theta}{2}$$

whence

$$\delta = 1.7 \pm .5 \times 10^{-6}$$

Comparing this with the theoretical value from the Lorentz equation

$$\delta = \frac{n\,e^2}{2\pi m}\cdot\frac{1}{\nu}$$

we take the density of calcite as 2.71 giving 81.5×10^{22} electrons per cubic centimeter, and place ν equal to the frequency of the molybdenum $K\alpha$ line, whence $\delta = 1.85 \times 10^{-6}$.

Determinations of the total reflection made with a second crystal gave results which checked with the preceding. Incidently it developed that the face of this second crystal had been ground at a slight angle to the reflecting planes, the crystal zero determined by the angle of total reflection differing from that obtained from the line spectra by an angle of 44′. This suggests an extension of the method of determining the refraction

from the shift of the spectral lines, for if the face of the crystal were ground at the proper angle to the reflecting planes, the shift would be increased enough to make accurate determinations possible.

[1] *Experimentelle Untersuchungen der Röntgenspektra*, Lund 1919.

[2] *J. Amer. Opt. Soc.*, May 1922.

[3] *Physic Rev., Ithaca*, July 1922.

THE COMPRESSIBILITY OF METALS AT HIGH PRESSURES

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Communicated October 17, 1922

This note briefly summarizes results which will be found in full detail in a forthcoming number of the PROCEEDINGS of the American Academy of Arts and Sciences. A considerable part of the expenses of the experiment was defrayed by a generous appropriation from the Rumford Fund of the American Academy.

A new method has been devised by which the difference of the linear compressibility of a solid in the form of a rod or wire and the linear compressibility of iron may be measured with an accuracy high enough to give the variation of compressibility with pressure or temperature. In addition, I have improved the method which I previously used[1] for the measurement of the absolute linear compressibility of iron, so that I have now measured the change of linear compressibility with pressure with some accuracy, and I have also obtained improved values for the change of linear compressibility with temperature. By combining these absolute measurements on iron with the measurements of the difference of compressibility the absolute linear compressibility of a number of metals has been obtained.

If the metal is equally compressible in all directions, the absolute volume compressibility may be calculated from the linear compressibility. The great majority of metals crystallize in the cubic system or in the hexagonal close packed arrangement of spheres, and for these the compressibility is equal in all directions. Table I gives the compressibility, χ (defined as $(\partial v/\partial p)_T$ where v is the volume of the amount of the metal that under standard conditions occupies 1 cc.) at atmospheric pressure at 30°, the pressure derivative of the compressibility, and the change of compressibility per degree rise of temperature for a number of such metals. The unit of pressure is the kilogram per square centimeter, the pressure range of the experiments was 12000 kg./cm.^{2}, and measure-

TABLE I

SUBSTANCE	$-\chi$ AT 30° $\times 10^7$	$-\frac{1}{\chi}\left(\frac{\partial\chi}{\partial p}\right)_T \times 10^6$	$+\frac{1}{\chi}\left(\frac{\partial\chi}{\partial \tau}\right)_p \times 10^4$	SUBSTANCE	$-\chi$ AT 30° $\times 10^7$	$-\frac{1}{\chi}\left(\frac{\partial\chi}{\partial p}\right)_T \times 10^6$	$+\frac{1}{\chi}\left(\frac{\partial\chi}{\partial \tau}\right)_p \times 10^4$
Al	13.43	0.74	5.4	Ni	5.29	0.802	2.5
Ca	56.97	1.66	6.0	Pd	5.28	0.80	1.3
Ce	35.74	1.06	0.4	Pt	3.60	1.00	3.7
Co	5.39	0.78	3.0	K	356.5	16.1	35.1
Cu	7.19	0.72	4.6	Ag	9.87	0.90	3.8
Ge	13.78	0.98	—2.3	Na	156.2	3.8	22.
Au	5.77	1.08		Sr	81.87	1.78	2.2
Fe	5.87	0.72	2.3	Ta	4.79	0.10	1.0
Pb	23.73	1.46	5.6	W	2.93	1.02	1.5
Li	86.92	2.24	7.2	Tl*	34.2	3.36	12.2
Mg	29.60	1.36	2.8	Ur†	9.66	0.52	—2.5
Mo	3.47	0.72	0.6				

* The crystal system of Tl has not yet been determined. The compressibility listed above was determined from a single specimen and is very materially higher than the value of other observers. It is therefore probable that the crystal system of Tl is not cubic.

† Crystal system of Ur not yet determined.

ments were made at two temperatures, 30° and 75°. The relation between volume and pressure of all these metals, except Na, K, and Tl, could be represented within the accuracy of the experiments by a second degree equation in the pressure. For the three exceptional metals, however, the relation is more complicated, and the coefficients listed in table I are only the initial values at atmospheric pressure. Of the above list of metals, the compressibility of Ge, Sr, and Ur seems never to have been measured, even at low pressures.

It is to be noticed that the order of magnitude of the relative change of compressibility with pressure or temperature is the same for all the metals. The compressibility of every metal decreases numerically with rising pressure, and in nearly every case increases with rising temperature. There are two apparent exceptions to this last statement in the cases of Ge and Ur. It is possible that the result found for Ur may be due to experimental error, but this explanation seems less likely in the case of Ge. This metal is known to be abnormal in other respects.

If the metal does not crystallize in a system whose compressibility is the same in all directions, then a complete description of the distortion under pressure requires that measurements be made of the linear compressibility in a number of directions of orientations known with respect to the crystal structure. This elaborate program I have not as yet been able to carry through, but I have been able to establish in several cases that there are very considerable differences of compressibility in different directions. These differences are shown by the fact that measurements on different

specimens of the same metal do not yield consistent results, unlike measurements on different specimens of cubic metals. If a non-cubic metal is measured first as a casting and then as an extruded rod, different results will nearly always be found. Results on two different castings, even, are not the same, unless the method of cooling the castings has been the same, so as to ensure the same average orientation of the crystal grains. By measuring the linear compressibility of non-cubic metals prepared in different ways, it is possible to set a lower limit to the differences of compressibility that would be found in different directions in a single crystal. This was the method adopted, and in table II are the results for several non-cubic metals. The values tabulated are the linear compressibility, k (defined as $(\partial l/\partial p)_\tau$, where l is a length parallel to one of the sides of a body of

TABLE II

SUBSTANCE		$-k$ AT 30° $\times 10^7$	$-\frac{1}{k}\left(\frac{\partial k}{\partial p}\right)_\tau \times 10^6$	$+\frac{1}{k}\left(\frac{\partial k}{\partial \tau}\right)_p \times 10^4$
Bi	cast	7.67	0.97	0.9
	extruded	11.78	1.82	3.7
Sb	cast	6.80	1.39	1.0
	cast	4.89	0.93	1.7
Cd	cast	4.72	1.15	5.6
	cast	6.51	1.23	7.4
	uni-crystalline	2.86	0.94	
Sn	extruded	6.51	1.12	6.6
	cast	5.67	0.71	4.7
Te	cast	—2.36		
Zn	cast	1.66		
		5.30		
		7.13		

metal which under standard conditions is a cube one centimeter on a side) at 30° at atmospheric pressure, and its pressure and temperature derivatives.

The three measurements on zinc were on pieces cut from the same original single crystal, oriented so as to give three directions mutually at right angles. For zinc, therefore, these numbers give a fair idea of the difference of compressibility in different directions in the same crystal. It is surprising that the difference is so large. The mean of the values listed above for three perpendicular directions is nearly the same as one-third the average cubic compressibility found by other observers, confirming the essential correctness of these measurements.

The negative compressibility of tellurium in a particular direction is perhaps unexpected, and so far as I know this is the first case found of this kind of behavior, although a somewhat analagous instance, the negative Poisson's ratio of pyrites, has been found by Voigt.

A number of theoretical considerations are suggested by these measurements of compressibility. In the first place, and most important, I have shown in the detailed paper by a dimensional argument that the correct order of magnitude for the compressibilities of all these metals may be found by taking over for the metal the same picture of the structure that has given such suggestive results when applied to the salts, namely the picture of the metal as composed of two interpenetrating lattices, one of positively charged atomic ions, and the other of negative electrons, the electron lattice in the case of the metal taking the place of the lattice of negatively charged atoms in the case of the salts, as for example the chlorine lattice in NaCl. The same conclusion as to the structure of a metal has been recently drawn by Kraus[2] on the basis of a large number of chemical facts.

The expression of Born[3] for the compressibility of salts, in which a repulsive potential between atoms as the inverse ninth power is found to give good values for the compressibility, cannot be extended to metals, for neither is the absolute value of the compressibility nor its variation with pressure such as would be given by this sort of a potential. There seems to be no simple repulsive potential that will account for the compressibility data for the various metals. It is possible, however, to represent the repulsive potential by an arbitrary function, and then to find the numerical value of several of its derivatives from experimental data. For those metals which crystallize face-centered cubic, and which therefore probably consist of interpenetrating lattices of ions and electrons of the type of NaCl, it is possible to calculate the grating constants, and to evaluate the first three derivatives of the repulsive potential in terms of the absolute dimensions of the crystal, the compressibility, and the change of compressibility with pressure. These derivatives for a number of metals are listed in the detailed paper. The sign of the derivatives is alternately positive and negative, and their successive absolute values differ by factors of the order of magnitude of the distance of separation of atomic centers. This is all as one would expect. To make more detailed use of these numerical values of the derivatives demands that more attention be paid to the details of the atomic structure than we have hitherto been able to do.

An effect recently discussed by Schottky,[4] namely a distortion of the atoms under pressure, I have shown by a detailed numerical examination to be of such a magnitude as to essentially modify the simple calculations of compressibility given by Born.

By the use of simple thermodynamics I have been able to modify the arguments of Born, which applied only at the absolute zero of temperature, so as to take account of the effect of temperature on the compressibility to be calculated in terms of the lattice structure. The new terms introduced by temperature are not large enough to seriously modify Born's argument.

Finally, with the new experimental data we are now in a position to calculate for the first time the changes in the specific heat of a metal at high pressures. It appears that the specific heat at constant volume decreases under pressure by an amount of the same order of magnitude as the change of volume under the same pressure, but in most cases by a factor several fold greater.

[1] Bridgman, P. W., *Proc. Amer. Acad., Boston*, **44**, 1909 (255–279), and **47**, 1911 (362–368).

[2] Kraus, C. A., *J. Amer. Chem. Soc., Easton, Pa.*, **44**, 1922 (1216–1238).

[3] Born, M., *Verh. Deut. Phys. Ges., Brauschweig*, **21**, 1919 (533–538).

[4] Schottky, W., *Phys. Zs., Leipzig*, **21**, 1920 (232–141).

A FURTHER NOTE ON THE MATHEMATICAL THEORY OF POPULATION GROWTH[1]

BY RAYMOND PEARL AND LOWELL J. REED

In an earlier paper in these PROCEEDINGS we[2] showed that the expression

$$y = \frac{be^{ax}}{1 + ce^{ax}} \tag{i}$$

gives an excellent fit to the known population growth of the United States since 1790. Since the first paper was published, we have extended and generalized our ideas on population growth with the results herein presented in skeleton outline. A more extended paper, giving a full discussion of our new results and of the pertinent literature is in press in Metron.

Considered generally, the curve

$$y = \frac{b}{e^{-ax} + c}$$

may be written

$$y = \frac{k}{1 + me^{ka'x}}, \tag{ii}$$

where

$$k = b/c,\ m = 1/c, \text{ and } ka' = -a.$$

Now the rate of change of y with respect to x is given by

$$\frac{dy}{dx} = -a'y\ (k-y)$$

or

$$\frac{\frac{dy}{dx}}{y\,(k-y)} = -a'. \tag{iii}$$

If y be the variable changing with time x (in our case population) equation (iii) amounts to the assumption that the time rate of change of y varies directly as y and as $(k-y)$, the constant k being the upper limit of growth, or, in other words, the value of the growing variable y at infinite time. Now since the rate of growth of y is dependent upon factors that vary with time, we may generalize (iii) by using $f\,(x)$ in place of $-a'$, $f\,(x)$ being some as yet undefined function of time.

Then

$$\frac{dy}{y\,(k-y)} = f(x)dx.$$

whence

$$\frac{k-y}{my} = e^{-k\int f(x)dx},$$

and

$$y = \frac{k}{1+me^{-k\int f(x)dx}} = \frac{k}{1+me^{F(x)}} \tag{iv}$$

where

$$F(x) = -k\int f(x)dx.$$

Then the assumption that the rate of growth of the dependent variable varies as (a) that variable, (b) a constant minus that variable, and (c) an arbitrary function of time, leads to equation (iv), which is of the same form as (i), except that ax has been replaced by $F(x)$. If now we assume that $f(x)$ may be represented by a Taylor series, we have

$$y = \frac{k}{1+me^{a_1x + a_2x^2 + a_3x^3 + \dots\dots\dots\dots + a_nx^n}} \tag{v}$$

If

$$a_2 = a_3 = a_4 = a_n = 0$$

then (v) becomes the same as (i).

If m becomes negative the curve becomes discontinuous at finite time. Since this cannot occur in the case of the growth of the organism or of populations, nor indeed so far as we are able to see, for any *phenomenal* changes with time, we shall restrict our further consideration of the equation to positive value only of m. Also since negative values of k would

give negative values of y, which in the case of population or individual growth are unthinkable, we shall limit k to positive values.

With these limitations as to the values of m and k we have the following general facts as to the form of (v). y can never be negative, i.e., less than zero, nor greater than k. Thus the complete curve always falls between the x axis and a line parallel to it at a distance k above it. Further we have the following relations:

$$\text{If} \qquad \begin{array}{ll} F(x) \doteq \infty & y=0 \\ F(x) \doteq -\infty & y=k \\ F(x) \doteq -0 & y=\dfrac{k}{1+m} \text{ from below} \\ F(x) \doteq +0 & y=\dfrac{k}{1+m} \text{ from below.} \end{array}$$

The maximum and minimum points of (v) occur where $\frac{dy}{dx} = 0$.

But $\frac{dy}{dx} = y(k-y) \;.\; F'(x)$,

therefore we have maximum and minimum points wherever $F'(x) = 0$. The fact that $\frac{dy}{dx} = 0$ when either $y = 0$ or $y - k = 0$ shows that the curve passes off to infinity asymptotic to the lines $y = 0$ and $y = k$.

The points of inflection of (v) are determined by the intersections of (v) with the curve.

$$y = \frac{k}{2} - \frac{k}{2}\,\frac{F''(x)}{[F'(x)]^2} \qquad \text{(vi)}$$

Since we are seldom justified in using over five arbitrary constants in any practical problem, we may limit equation (v) still further by stopping at the third power of x. This gives the equation

$$y = \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}}. \qquad \text{(vii)}$$

If a_n is positive the curve of equation (v) is reversed and becomes asymptotic to a line $A\,B$, at $x = -\infty$ and to the x axis at $x = +\infty$. Thus in equation (vii) a_3 negative is a case of growth, and a_3 positive is a case of decay.

Equation (vii) has several special forms that are of interest, among them being a form similar in shape to the autocatalytic curve (i.e., with no maximum or minimum points and only one point of inflection) except that it is free from the two restrictive features mentioned in our first

paper, that is, location of the point of inflection in the middle and symmetry of the two limbs of the curve. Asymmetrical or skew curves of this sort can only arise when the equation, $F'(x) = 0$, has no real roots. While any odd value of n may yield this form of curve the simplest equation that will do it is that in which $n = 3$, so that the equation of this curve becomes that of (vii).

Having determined that the growth within any one epoch or cycle may be approximately represented by equation (i), or more accurately by (vii), the next question is that of treating several epochs or cycles. Theoretically, some form of (v) may be found by sufficient labor in the adjustment of constants so that one equation with say 5 or 7 constants would describe a long history of growth involving several cycles. Practically, however, we have found it easier and just as satisfactory in other respects to treat each cycle by itself. Since the cycles of any case of growth are additive, we may use for any single cycle the equation

$$y = d + \frac{k}{1 + me^{ka'x}} \tag{viii}$$

or more generally

$$y = d + \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}} \tag{ix}$$

where in both of these forms d represents the total growth attained in all the previous cycles. The term d is therefore the lower asmptote of the cycle of growth under consideration and $d + k$ is its upper asymptote.

In treating any two adjacent cycles, it should be noted that the lower asymptote of the second cycle is frequently below the upper asymptote of the first cycle, due to the fact that the second cycle is often started before the first one has had time to reach its natural level. This for instance would be the case where a population entered upon an industrial era before the country had reached the limit of population possible under purely agricultural conditions.

The theory presented in this paper has been found to be entirely successful in fitting the population growth of many different countries, and in a subsequent publication this fact will be demonstrated with examples.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, The Johns Hopkins University, No. 81.

[2] Pearl, R. and Reed, L. J., "On the Rate of Growth of the Population of the United States Since 1790 and Its Mathematical Representation." PROC. NAT. ACAD. SCI., Vol. 6, pp. 275–288, 1920.

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